

Growth description of p th means of the Green potential in the unit ball

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Abstract

We describe the growth of p th means, $1 < p < \frac{2n-1}{2(n-1)}$, of the invariant Green potential in the unit ball in \mathbb{C}^n in terms of smoothness properties of a measure. In particular, a criterion of boundedness of p th means of the potential is obtained, a result of M. Stoll is generalized.

Keywords: Green potential, unit ball, invariant Laplacian, M -subharmonic function, Riesz measure

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Firstly, we introduce some definitions and basic notation ([13]). For $n \in \mathbb{N}$, let \mathbb{C}^n denote the n -dimensional complex space with the inner product

$$\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j, \quad z, w \in \mathbb{C}^n.$$

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball and $S = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere, where $|z| = \sqrt{\langle z, z \rangle}$.

For $z, w \in B$, define the *involutive automorphism* φ_w of the unit ball B given by

$$\varphi_w(z) = \frac{w - P_w z - (1 - |w|^2)^{1/2} Q_w z}{1 - \langle z, w \rangle}$$

where $P_0 z = 0$, $P_w z = \frac{\langle z, w \rangle}{|w|^2} w$, $w \neq 0$, is the orthogonal projection of \mathbb{C}^n onto the subspace generated by w and $Q_w = I - P_w$.

The invariant Laplacian $\tilde{\Delta}$ on B is defined by

$$\tilde{\Delta}f(a) = \Delta(f \circ \varphi_a)(0),$$

where $f \in C^2(B)$, Δ is the ordinary Laplacian. It is known that $\tilde{\Delta}$ is invariant w.r.t. any holomorphic automorphism of B ([9, Chap.4], [13]).

The Green's function for the invariant Laplacian is defined by $G(z, w) = g(\varphi_w(z))$, where $g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt$ ([13, Chap.6.2]).

If μ is a nonnegative Borel measure on B , the function G_μ defined by

$$G_\mu(z) = \int_B G(z, w) d\mu(w)$$

is called the (*invariant*) *Green potential* of μ , provided $G_\mu \not\equiv +\infty$. It is known that ([13, Chap.6.4]) the last condition is equivalent to

$$\int_B (1 - |w|^2)^n d\mu(w) < \infty. \quad (1)$$

The concept of the invariant Laplacian naturally implies the following definition ([13], [15]). A function u on B is called *\mathcal{M} -subharmonic* if it is upper semicontinuous and $\tilde{\Delta}u \geq 0$ in the sense of distributions. In particular, $-G_\mu$ is *\mathcal{M} -subharmonic*. A function u on B is called *\mathcal{M} -harmonic* if $u \in C^2(B)$ and $\tilde{\Delta}u = 0$. Unlike the class of plurisubharmonic functions a counterpart of the Riesz decomposition theorem holds for the class of *\mathcal{M} -subharmonic* functions (see [15], [13]). Due to this theorem, an *\mathcal{M} -subharmonic* function u with the norm $\|u(r\xi)\|_{L^1(S)}$ uniformly bounded on $[0, 1)$ can be represented as the difference of an *\mathcal{M} -harmonic* function and the Green potential of a nonnegative measure satisfying (1). Thus investigations of the Green potentials are very important in studying the whole class of *\mathcal{M} -subharmonic* functions. Note that in the case $n = 1$ the classes of *\mathcal{M} -subharmonic* functions and subharmonic functions coincide.

Let $0 < p < \infty$, u be a measurable function locally integrable on B . We define

$$m_p(r, u) = \left(\int_S |u(r\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}}$$

where $d\sigma$ is the Lebesgue measure on S normalized so that $\sigma(S) = 1$.

The aim of the paper is to *describe the growth (decrease) of $m_p(r, G_\mu)$ in terms of properties of the measure μ* . In the case $n = 1$, $p = 2$, this is closely

connected to the question of A. Zygmund, who asked on criterion of boundedness of $m_2(r, \log |B|)$, where B is a Blaschke product. G. MacLane and L. Rubel in [8] answered his question. Corollary 2 below is a boundedness criterion for $m_p(r, u)$, $1 < p < \frac{2n-1}{2(n-1)}$.

In the case $n > 1$, sharp estimates of the growth rate of $m_p(r, G_\mu)$ for the whole class of Borel measures satisfying (1) are proved by M. Stoll in [11].

Theorem A ([11]). *Let G_μ be the Green potential on B .*

(1) *If $1 \leq p < \frac{2n-1}{2(n-1)}$, then*

$$\lim_{r \rightarrow 1-} (1 - r^2)^{n(1-1/p)} m_p(r, G_\mu) = 0. \quad (2)$$

(2) *If $n \geq 2$ and $\frac{2n-1}{2(n-1)} \leq p < \frac{2n-1}{2n-3}$, then*

$$\liminf_{r \rightarrow 1-} (1 - r^2)^{n(1-1/p)} m_p(r, G_\mu) = 0. \quad (3)$$

The examples in [11] show that the estimates are the best possible in some sense. We note that similar results for the Green potentials in the unit ball in \mathbb{R}^n are established earlier in [4] (cf. [10]). For the recent development in the real case we address to the book [14]. The case $n = 1$ is studied much more deeper, see e.g. [6], [7].

Remark 1. *Under additional restrictions on the measure μ estimates of growth rate $m_p(r, G_\mu)$ can be improved.*

Theorem B ([12, Theorem 3.2]). *Let μ be a Borel measure on B satisfying*

$$\int_B (1 - |w|^2)^\beta d\mu(w) < \infty$$

for some real $\beta \leq n$.

(1) *If $1 \leq p < \frac{2n-1}{2(n-1)}$ and $-n(1 - 1/p) < \beta \leq n$, then*

$$\lim_{r \rightarrow 1-} (1 - r^2)^{\beta - n/p} m_p(r, G_\mu) = 0.$$

(2) *If $n \geq 2$ and $\frac{2n-1}{2(n-1)} \leq p < \frac{2n-1}{2n-3}$ and $-n(1 - 1/p) < \beta \leq n$, then*

$$\liminf_{r \rightarrow 1-} (1 - r^2)^{\beta - n/p} m_p(r, G_\mu) = 0.$$

Remark 2. *Theorem B implies Theorem A for $\beta = n$.*

Remark 3. *It follows from results of [15] (see also [13]) that*

$$m_1(r, G_\mu) = o(1) \quad r \rightarrow 1-.$$

So we omit the case $p = 1$.

Remark 4. *It is shown in [11, Example 2] that for each $n \geq 2$ there exists a discrete measure μ satisfying (1) such that*

$$\limsup_{r \rightarrow 1-} (1 - r^2)^{n(1-1/p)} m_p(r, G_\mu) = \infty$$

for all $p \geq \frac{2n-1}{2(n-1)}$.

In view of Remarks 3 and 4, we confine to the case $1 < p < \frac{2n-1}{2(n-1)}$.

Theorem A gives the maximal growth rate of the p th mean of the Green potentials, but does not take into account particular properties of a measure μ . It is appeared that smoothness properties of so called *complete measure* (in the sense of Grishin [5], [1], [2]) or the *related measure* (see [3]) of a subharmonic function allow to describe its growth. Here we just note that in the case when $n = 1$ and $u = -G_\mu$, the complete measure $\lambda = \lambda_u$ of u is the weighted Riesz measure $d\lambda(z) = (1 - |z|)d\mu(z)$. In particular, results from [2] imply

Theorem C ([2]). *Let $\gamma \in (0, 1]$, $p \in (1, \infty)$, $n = 1$, and μ be a Borel measure satisfying (1). Let λ be defined as above. Necessary and sufficient that*

$$m_p(r, G_\mu) = O((1 - r)^{\gamma-1}), \quad r \rightarrow 1-,$$

hold is that

$$\int_0^{2\pi} \lambda^p(\{\rho e^{i\theta} \in B : \rho \geq 1 - \delta, |\theta - \varphi| \leq \pi\delta\}) d\varphi = O(\delta^{p\gamma}), \quad 0 < \delta < 1.$$

Define for $a, b \in \bar{B}$ the *anisotropic metric* on S by $d(a, b) = |1 - \langle a, b \rangle|^{1/2}$ ([9, Chap.5.1]).

For $\xi \in S$ and $\delta > 0$ we denote

$$C(\xi, \delta) = \{z \in B : d(z, \xi) < \delta^{1/2}\}, \quad D(\xi, \delta) = \{z \in B : d(z, \xi) < \delta\},$$

and $d\lambda(z) = (1 - |z|)^n d\mu(z)$.

The following theorem is our main result. As we can see later, it generalizes Theorem A(1) and Theorem B(1).

Theorem 1. *Let $n > 1$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \leq \gamma < 2n$, μ be a Borel measure satisfying (1). Then*

$$m_p(r, G_\mu) = O((1 - r)^{\gamma-n}), \quad r \uparrow 1 \quad (4)$$

holds if and only if

$$\left(\int_S \lambda^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = O(\delta^\gamma), \quad 0 < \delta < 1. \quad (5)$$

As a corollary we obtain a criterion of boundedness of the invariant Green potential.

Corollary 2. *Let $n > 1$, $1 < p < \frac{2n-1}{2(n-1)}$, μ be a Borel measure satisfying (1). Then*

$$m_p(r, G_\mu) = O(1), \quad 0 < r < 1$$

if and only if

$$\left(\int_S \lambda^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = O(\delta^n), \quad 0 < \delta < 1. \quad (6)$$

Remark 5. *For $\gamma \in (n, 2n)$, Theorem 1 gives necessary and sufficient conditions for decrease of the Green potential.*

Example 3. *If μ is the Lebesgue measure on B , then*

$$\lambda^p(C(\xi, \delta)) = O(\delta^{n+1}) \quad (7)$$

i.e. the assumption (5) holds with $\gamma = n + 1$, thus $m_p(r, G_\mu) = O(1 - r)$ as $r \rightarrow 1-$ and $n > 1$. The latter relation is valid in the case $n = 1$ as well, which can be checked directly.

The estimate (7) follows from the next remarks. Firstly, the radial projection of $C(\xi, \delta)$ on S has $(2n-1)$ -dimensional measure $\sigma_\delta = c\delta^n$ ([9, Prop. 5.1.4]). Secondly, by the definition, $C(\xi, \delta) \subset \{z \in B : |z| \geq 1 - \delta\}$.

It is sometimes suitable to have an “o”-analog of Theorem 1.

Theorem 4. *Let $n > 1$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \leq \gamma < 2n$, μ be a Borel measure satisfying (1). Then*

$$m_p(r, G_\mu) = o((1-r)^{\gamma-n}), \quad r \rightarrow 1- \quad (8)$$

holds if and only if

$$\left(\int_S \lambda^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = o(\delta^\gamma), \quad \delta \rightarrow 0+. \quad (9)$$

The following elementary proposition is useful.

Proposition 5. *Let $n \in \mathbb{N}$, ν be a finite Borel measure on B . Then*

$$\left(\int_S \nu^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = o(\delta^{\frac{n}{p}}), \quad \delta \rightarrow 0+. \quad (10)$$

Remark 6. *Theorem 4 and Proposition 5 imply Theorem A(1) as a corollary for $p > 1$. In Section 2 we show that Theorem 4 and Proposition 5 imply Theorem B(1) as well.*

In the sequel, the symbol c stands for positive constants which depend on the parameters indicated in the parentheses, $a \asymp b$ means that there are positive constants c' and c'' such that $c'a < b < c''a$ holds.

1 Auxiliary results

The following lemma gives some basic properties of g which will be needed later.

Lemma A ([13]). *Let $0 < \delta < \frac{1}{2}$ be fixed. Then g satisfies the following relations:*

$$\begin{aligned} g(z) &\geq \frac{n+1}{4n^2}(1-|z|^2)^n, \quad z \in B, \\ g(z) &\leq c(\delta)(1-|z|^2)^n, \quad z \in B, |z| \geq \delta, \end{aligned} \quad (11)$$

where $c(\delta)$ is a positive constant. Furthermore, if $n > 1$ then

$$g(z) \asymp |z|^{-2n+2}, \quad |z| \leq \delta. \quad (12)$$

We need the following multidimensional generalization of Lemma 1 from [2].

Lemma 1. *Let ν be a finite positive Borel measure on S , $0 < \delta < \frac{1}{2}$, and $p \geq 1$. Then*

$$\int_S \nu^{p-1}(D(\xi, \delta)) d\nu(\xi) \leq \frac{N^p}{\delta^{2n}} \int_S \nu^p(D(\xi, \delta)) d\sigma(\xi),$$

where N is a positive constant independent of p and δ .

Proof of the lemma. First, we prove the statement for $p = 1$. Since ([9, Prop. 5.1.4]) $\sigma(D(\xi, \delta)) \asymp \delta^{2n}$, one has

$$\int_S d\nu(\xi) \leq \frac{c}{\delta^{2n}} \int_S d\nu(\xi) \int_{D(\xi, \delta)} d\sigma(t). \quad (13)$$

Let $\Theta: \Pi \rightarrow S$ be the spherical coordinates on the unit sphere, where $\Pi = [0, \pi]^{2n-2} \times [0, 2\pi)$. Since Θ is periodic in each variable, we consider Θ on \mathbb{R}^{2n-1} . We set $\Pi' = [-\frac{\pi}{2}, \frac{3\pi}{2}]^{2n-2} \times [-\pi, 3\pi)$. Then, using Fubini's theorem and the periodicity of the Jacobian $\det \Theta'$, we deduce

$$\begin{aligned} \int_S d\nu(\xi) \int_{D(\xi, \delta)} d\sigma(t) &= \int_{\Pi} d\nu(\Theta(x)) \int_{\substack{d(\Theta(x), \Theta(y)) < \delta \\ |x-y| < \frac{\pi}{2}}} |\det \Theta'(y)| dy \\ &\leq \int_{\Pi'} dy \int_{\substack{d(\Theta(x), \Theta(y)) < \delta \\ |x-y| < \frac{\pi}{2}}} |\det \Theta'(y)| d\nu(\Theta(x)) \\ &= 2^{2n-1} \int_{\Pi} dy \int_{\substack{d(\Theta(x), \Theta(y)) < \delta \\ |x-y| < \frac{\pi}{2}}} |\det \Theta'(y)| d\nu(\Theta(x)) \\ &= 2^{2n-1} \int_S d\sigma(t) \int_{D(t, \delta)} d\nu(\xi) = 2^{2n-1} \int_S \nu(D(t, \delta)) d\sigma(t). \end{aligned}$$

Substituting this estimate into (13), we obtain the statement of the lemma in the case $p = 1$ with $N = c2^{2n-1}$.

Let now $p > 1$. We define $d\nu_1(\xi) = \nu^{p-1}(D(\xi, \delta)) d\nu(\xi)$. Then applying

the statement of the lemma for $p = 1$ we get

$$\begin{aligned}
\int_S \nu^{p-1}(D(\xi, \delta)) d\nu(\xi) &= \int_S d\nu_1(\xi) \leq 2^{2n-1} c \int_S \frac{\nu_1(D(t, \delta))}{\delta^{2n}} d\sigma(t) \\
&= \frac{2^{2n-1} c}{\delta^{2n}} \int_S \left(\int_{D(t, \delta)} \nu^{p-1}(D(\xi, \delta)) d\nu(\xi) \right) d\sigma(t) \\
&\leq \frac{2^{2n-1} c}{\delta^{2n}} \int_S \nu^{p-1}(D(t, 2\delta)) \nu(D(t, \delta)) d\sigma(t) \\
&\leq \frac{2^{2n-1} c}{\delta^{2n}} \int_S \nu^p(D(t, 2\delta)) d\sigma(t). \tag{14}
\end{aligned}$$

Let $\{t_1, \dots, t_N\} \subset S$ be a finite δ -net for $D(e_1, 2\delta)$ where $e_1 = (1, 0, \dots, 0)$ and N depends on n only, i.e. $\bigcup_{k=1}^N D(t_k, \delta) \supset D(e_1, 2\delta)$. Then t_k can be represented in the form $t_k = \tau_k(e_1)$, where $\tau_k \in U(n)$, $k \in \{1, \dots, N\}$, are unitary transformations of \mathbb{C}^n . Taking into account that the measure σ is invariant w.r.t. the elements of $U(n)$, we deduce

$$\begin{aligned}
\int_S \nu^p(D(t, 2\delta)) d\nu(t) &\leq \int_S \nu^p \left(\bigcup_{k=1}^N D(\tau_k(t), \delta) \right) d\sigma(t) \\
&\leq N^{p-1} \sum_{k=1}^N \int_S \nu^p(D(\tau_k(t), \delta)) d\sigma(t) = N^{p-1} \sum_{k=1}^N \int_S \nu^p(D(t, \delta)) d\sigma(t) \\
&= N^p \int_S \nu^p(D(t, \delta)) d\sigma(t).
\end{aligned}$$

Taking into account (14) we finish the proof of the lemma. \square

2 Proofs of the main results

Proof of Theorem 1. Sufficiency. Denote

$$B^*\left(z, \frac{1}{4}\right) = \left\{ w \in B : |\varphi_w(z)| < \frac{1}{4} \right\}.$$

Let us estimate the absolute values of

$$u_1(z) := \int_{B^*\left(z, \frac{1}{4}\right)} G(z, w) d\mu(w) \quad \text{and} \quad u_2(z) := \int_{B \setminus B^*\left(z, \frac{1}{4}\right)} G(z, w) d\mu(w).$$

We start with u_1 . By definition

$$0 \leq u_1(z) = \int_{B^*(z, \frac{1}{4})} G(z, w) d\mu(w) = \int_{B^*(z, \frac{1}{4})} g(\varphi_w(z)) d\mu(w).$$

By (12) we have $g(z) \geq c|z|^{-2n+2}$ for $|z| \leq \frac{1}{4}$ and some positive constant c . Thus,

$$|u_1(z)| \leq c \int_{B^*(z, \frac{1}{4})} |\varphi_w(z)|^{-2n+2} d\mu(w).$$

Denote $z = r\xi$, where $r = |z|$, $\frac{1}{2} < r < 1$ and $w = |w|\eta$, $\xi, \eta \in S$. Let

$$K(z, \sigma_1, \sigma_2) = \{w \in B : |r - |w|| \leq \sigma_1, d(\xi, \eta) \leq \sigma_2\}.$$

Now let us proof the inclusion

$$B^*\left(z, \frac{1}{4}\right) \subset K\left(z, c_1(1-r), c_2(1-r)^{\frac{1}{2}}\right) \quad (15)$$

where c_1, c_2 are positive constants. Since ([13, p.11]) for $w \in B^*(z, \frac{1}{4})$

$$\left(1 - \frac{(1 - |w|^2)(1 - r^2)}{|1 - \langle z, w \rangle|^2}\right)^{\frac{1}{2}} = |\varphi_w(z)| < \frac{1}{4}, \quad (16)$$

$$A(z) := \left\{w \in B : \left(1 - \frac{(1 - |w|^2)(1 - r^2)}{(1 - r|w|)^2}\right)^{\frac{1}{2}} < \frac{1}{4}\right\} \supset B^*\left(z, \frac{1}{4}\right).$$

In order to find c_1 it is enough to check that

$$\partial A(z) \subset \overline{K}(z, c_1(1-r), c_2(1-r)^{\frac{1}{2}}).$$

For $z \in \partial A(z)$ one has $\frac{(1-|w|^2)(1-r^2)}{(1-r|w|)^2} = \frac{15}{16}$. Solving this equation, we find $|w_1| = \frac{4r+1}{r+4}$, $|w_2| = \frac{4r-1}{4-r}$. So

$$\begin{aligned} |r - |w_1|| &= |w_1| - r = \frac{1+r}{r+4}(1-r) < \frac{2}{5}(1-r), \\ |r - |w_2|| &= r - |w_2| = \frac{1+r}{4-r}(1-r) < \frac{2}{3}(1-r). \end{aligned}$$

Thus, $|r - |w|| < \frac{2}{3}(1 - r)$ for $w \in B^*(z, \frac{1}{4})$. Then, to estimate c_2 we deduce

$$\begin{aligned}
d^2(\xi, \eta) &= |1 - \langle \xi, \eta \rangle| < \frac{1}{r|w|} |1 - r|w||\langle \xi, \eta \rangle| \\
&< \frac{1}{r|w|} \left(\frac{16}{15} (1 - |w|^2)(1 - r^2) \right)^{\frac{1}{2}} \\
&< \frac{1}{r(-2/3 + 5r/3)} \left(\frac{16}{15} \left(1 - \left(-\frac{2}{3} + \frac{5}{3}r \right)^2 \right) (1 - r^2) \right)^{\frac{1}{2}} \\
&= \frac{4}{\sqrt{3}r(5r - 2)} ((5r + 1)(1 + r))^{\frac{1}{2}} (1 - r) < 32(1 - r),
\end{aligned}$$

where $\frac{1}{2} < r < 1$. So (15) holds with $c_1 = \frac{2}{3}$ and $c_2 = 4\sqrt{2}$. We denote

$$\begin{aligned}
K(z) &:= K \left(z, \frac{2}{3}(1 - r), 4\sqrt{2}(1 - r)^{\frac{1}{2}} \right), \\
\tilde{K}(z) &:= K \left(z, \frac{2}{3}(1 - r), 8\sqrt{2}(1 - r)^{\frac{1}{2}} \right).
\end{aligned}$$

Hölder's inequality and inclusion (15) imply

$$\begin{aligned}
I_1 &:= \int_S |u_1(r\xi)|^p d\sigma(\xi) \\
&\leq c_3 \int_S \left(\int_{B^*(r\xi, \frac{1}{4})} |\varphi_w(r\xi)|^{-2n+2} d\mu(w) \right)^p d\sigma(\xi) \\
&\leq c_3 \int_S \int_{B^*(r\xi, \frac{1}{4})} |\varphi_w(r\xi)|^{-p(2n-2)} d\mu(w) \mu^{p-1} \left(B^* \left(r\xi, \frac{1}{4} \right) \right) d\sigma(\xi) \\
&\leq c_3 \int_S \int_{K(r\xi)} \frac{d\mu(w)}{|\varphi_w(r\xi)|^{p(2n-2)}} \mu^{p-1}(K(r\xi)) d\sigma(\xi) \\
&\leq c_3 \int_S \int_{K(r\xi)} \frac{\mu^{p-1}(\tilde{K}(r\xi))}{|\varphi_w(r\xi)|^{p(2n-2)}} d\mu(|w|\eta) d\sigma(\xi)
\end{aligned}$$

where $c_3 = c_3(p)$. Then, by Fubini's theorem (cf. the proof of Lemma 1)

we deduce ($z = r\xi$, $w = |w|\eta$)

$$\begin{aligned}
I_1 &\leq c_4(n, p) \iint_{\substack{\eta \in S \\ ||w|-r| < \frac{2}{3}(1-r) \\ d(\xi, \eta) < 4\sqrt{2}(1-r)^{1/2}}} \frac{\mu^{p-1}(\tilde{K}(r\eta))}{|\varphi_w(r\xi)|^{p(2n-2)}} d\mu(|w|\eta) d\sigma(\xi) \\
&\leq c_4(p, n) \int_{||w|-r| < \frac{2}{3}(1-r)} \mu^{p-1}(\tilde{K}(r\eta)) \int_S \frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{p(2n-2)}} d\mu(w). \quad (17)
\end{aligned}$$

Applying subsequently (16), (11) and Lemma 5 ([11]), we obtain that for $1 < p < \frac{2n-1}{2(n-1)}$

$$\begin{aligned}
&\int_S \frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{p(2n-2)}} = \int_S \frac{d\sigma(\xi)}{|\varphi_{r\xi}(w)|^{p(2n-2)}} \\
&\leq \int_S g^p(\varphi_{r\xi}(w)) d\sigma(\xi) \leq \frac{c_5(1-|w|^2)^{np}}{(1-r^2)^{n(p-1)}}, \quad \frac{1}{2} < r < 1.
\end{aligned}$$

Substituting the estimate of the inner integral into (17) we get

$$\begin{aligned}
I_1 &\leq c_4 \int_{||w|-r| < \frac{2}{3}(1-r)} \frac{c_5(1-|w|^2)^{np}}{(1-r^2)^{n(p-1)}} \mu^{p-1}(\tilde{K}(r\eta)) d\mu(|w|\eta) \\
&\leq c_6(1-r)^n \int_{||w|-r| < \frac{2}{3}(1-r)} \mu^{p-1}(\tilde{K}(r\eta)) d\mu(|w|\eta). \quad (18)
\end{aligned}$$

To obtain the final estimate of I_1 , for a fixed $r \in (\frac{1}{2}, 1)$, we define the measure ν_1 on the balls by

$$\nu_1(D(\eta, t)) = \lambda\left(\left\{\rho\zeta \in B : |\rho - r| < \frac{2}{3}(1-r), d(\zeta, \eta) < t\right\}\right).$$

It can be expanded to the family of all Borel sets on B in the standard way. It is clear that

$$\nu_1(D(\eta, t)) \asymp (1-r)^n \mu\left(\left\{\rho\zeta \in B : |\rho - r| < \frac{2}{3}(1-r), d(\zeta, \eta) < t\right\}\right).$$

By using of (18) and Lemma 1 we get

$$\begin{aligned}
I_1 &\leq \frac{c_7}{(1-r)^{n(p-1)}} \int_{||w|-r|<\frac{2}{3}(1-r)} \lambda^{p-1} \left(\tilde{K}(r\eta) \right) d\lambda(|w|\eta) \\
&= \frac{c_7}{(1-r)^{n(p-1)}} \int_S \nu_1^{p-1} \left(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{2}}) \right) d\nu_1(\eta) \\
&\leq \frac{c_7 N^p}{(128)^n (1-r)^{np}} \int_S \nu_1^p \left(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{2}}) \right) d\sigma(\eta) \\
&= \frac{c_8(n, p)}{(1-r)^{np}} \int_S \lambda^p \left(\tilde{K}(r\eta) \right) d\sigma(\eta).
\end{aligned}$$

Note that if $\rho\zeta \in \tilde{K}(r\eta)$ then

$$|1 - \langle \rho\zeta, \eta \rangle| \leq |1 - \langle \zeta, \eta \rangle| + (1 - \rho) |\langle \zeta, \eta \rangle| \leq (4c_2^2 + c_1 + 1)(1 - r). \quad (19)$$

Hence, by the assumption of the theorem

$$\begin{aligned}
I_1 &\leq c_8(1-r)^{-np} \int_S \lambda^p \left(C(\eta, (4c_2^2 + c_1 + 1)(1-r)) \right) d\sigma(\eta) \\
&\leq c_9(1-r)^{p(\gamma-n)}. \quad (20)
\end{aligned}$$

Let us estimate

$$u_2(z) = \int_B G(z, w)(1 - |w|)^{-n} d\tilde{\lambda}(w)$$

where $d\tilde{\lambda}(w) = (1 - |w|)^n \chi_{B \setminus B^*(z, \frac{1}{4})}(w) d\mu(w)$, χ_E is the characteristic function of a set E . We may assume that $|z| \geq \frac{1}{2}$.

We denote

$$E_k = E_k(z) = \left\{ w \in B : \left| 1 - \left\langle \frac{z}{|z|}, w \right\rangle \right| < 2^{k+1}(1 - |z|) \right\}, \quad k \in \mathbb{Z}_+.$$

Then for $w \in E_{k+1}(z) \setminus E_k(z)$, $\frac{1}{2} \leq |z| < 1$

$$|1 - \langle z, w \rangle| \geq |z| \left| 1 - \left\langle \frac{z}{|z|}, w \right\rangle \right| - (1 - |z|) \geq (|z|2^{k+1} - 1)(1 - |z|).$$

Combining Lemma A with the equality in (16) for $z \in B$ such that $|z| \geq \frac{1}{2}$ we get that $0 \leq G(z, w) \leq c_{10} \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z, w \rangle|^2} \right)^n$ holds. So

$$\begin{aligned}
|u_2(z)| &\leq c_{10} \int_B \left(\frac{(1+|w|)(1-|z|^2)}{|1-\langle z, w \rangle|^2} \right)^n d\tilde{\lambda}(w) \\
&\leq \sum_{k=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor} c_{10} \int_{E_{k+1} \setminus E_k} \left(\frac{(1+|w|)(1-|z|^2)}{(|z|2^{k+1}-1)^2(1-|z|^2)} \right)^n d\tilde{\lambda}(w) \\
&\quad + c_{10} \int_{E_1} \left(\frac{(1+|w|)(1-|z|^2)}{(1-|z|^2)^2} \right)^n d\tilde{\lambda}(w) \\
&\leq \sum_{k=1}^{\infty} \int_{E_{k+1} \setminus E_k} \frac{4^n c_{10}}{(2^{2(k-1)}(1-|z|))^n} d\tilde{\lambda}(w) + \int_{E_1} \frac{4^n c_{10}}{(1-|z|)^n} d\tilde{\lambda}(w) \\
&\leq \frac{4^n c_{10}}{(1-|z|)^n} \left(\sum_{k=1}^{\infty} \frac{\tilde{\lambda}(E_{k+1})}{2^{2n(k-1)}} + \tilde{\lambda}(E_1) \right) \leq \frac{4^n c_{10}}{(1-|z|)^n} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}(E_k)}{2^{2n(k-2)}}.
\end{aligned}$$

Fix any $\alpha \in (\frac{\gamma}{n}, 2)$. By Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned}
|u_2(z)|^p &\leq \frac{4^{np} c_{10}^p}{(1-|z|)^{np}} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}^p(E_k)}{2^{\alpha np(k-2)}} \left(\sum_{k=1}^{\infty} \frac{1}{2^{(2-\alpha)nq(k-2)}} \right)^{p/q} \\
&= \frac{4^{np} c_{10}^p}{(1-|z|)^{np}} \frac{2^{4np}}{(2^{2-\alpha)nq} - 1)^{p/q}} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}^p(E_k)}{2^{\alpha npk}} = \frac{c_{11}(n, p, \alpha)}{(1-|z|)^{np}} \sum_{k=1}^{\infty} \frac{\tilde{\lambda}^p(E_k)}{2^{\alpha npk}}. \quad (21)
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_S |u_2(r\xi)|^p d\sigma(\xi) &\leq \frac{c_{11}}{(1-r)^{np}} \sum_{k=1}^{\infty} \int_S \frac{\tilde{\lambda}^p(E_k(r\xi))}{2^{\alpha npk}} d\sigma(\xi) \\
&= \frac{c_{11}}{(1-r)^{np}} \sum_{k=1}^{\infty} \frac{1}{2^{\alpha npk}} \int_S \tilde{\lambda}^p \left(C \left(\frac{z}{|z|}, 2^{k+1}(1-r) \right) \right) d\sigma(\xi) \\
&\leq \frac{c_{12}}{(1-r)^{np}} \sum_{k=1}^{\infty} \frac{2^{p\gamma(k+1)}(1-r)^{\gamma p}}{2^{\alpha npk}} \\
&= \frac{c_{12}}{(1-r)^{p(n-\gamma)}} \frac{2^{p\gamma}}{2^{p(\alpha n-\gamma)} - 1} = \frac{c_{13}(n, p, \gamma)}{(1-r)^{p(n-\gamma)}}.
\end{aligned}$$

The latter inequality together with (20) completes the proof of the sufficiency.

Necessity. By Lemma A

$$\begin{aligned}
G_\mu(r\xi) &= \int_B g(\varphi_w(r\xi)) d\mu(w) \geq \int_B \frac{n+1}{4n^2} \frac{(1-|w|^2)^n (1-r^2)^n}{|1-\langle r\xi, w \rangle|^{2n}} d\mu(w) \\
&\geq \int_{C(\xi, 1-r)} \frac{n+1}{4n^2} \frac{(1-|w|^2)^n (1-r^2)^n}{|1-\langle r\xi, w \rangle|^{2n}} d\mu(w) \\
&= \int_{C(\xi, 1-r)} \frac{n+1}{4n^2} \frac{(1+|w|)^n (1-r^2)^n}{|1-\langle r\xi, w \rangle|^{2n}} d\lambda(w).
\end{aligned}$$

Since for $w \in C(\xi, 1-r)$

$$|1 - \langle z, w \rangle| \leq \left| 1 - \left\langle \frac{z}{|z|}, w \right\rangle \right| + \left| \left\langle \frac{z}{|z|} - z, w \right\rangle \right| \leq 2(1 - |z|),$$

we have

$$|G_\mu(r\xi)| \geq \frac{n+1}{4^{n+1}n^2} \frac{\lambda(C(\xi, 1-r))}{(1-r)^n}.$$

From the assumption of the theorem it follows that

$$\begin{aligned}
&\left(\frac{n+1}{2^{2(n+1)}n^2} \right)^p \int_S \frac{\lambda^p(C(\xi, 1-r))}{(1-r)^{np}} d\sigma(\xi) \\
&\leq \int_S |G_\mu(r\xi)|^p d\sigma(\xi) \leq c_{13}^p (1-r)^{p(\gamma-n)}.
\end{aligned}$$

Thus

$$\int_S \lambda^p(C(\xi, 1-r)) d\sigma(\xi) \leq c_{13}^p (1-r)^{p\gamma}, \quad 0 < r < 1.$$

□

Proof of Theorem 4. The proof of the *necessity* literally repeats that of Theorem 1.

In the *sufficiency* part the estimate of u_1 is quite similar. In order to estimate u_2 we note that, by the definition of $E_k(z)$,

$$1 - |w| \leq \left| 1 - \left\langle \frac{z}{|z|}, w \right\rangle \right| \leq 2^{k+1}(1 - |z|) \leq 2\sqrt{1 - |z|},$$

for $w \in E_k(z)$, $1 \leq k \leq \frac{1}{2} \log_2 \frac{1}{1-|z|}$. Thus (9) implies as $r \rightarrow 1-$

$$\sum_{k=1}^{\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]} \int_S \tilde{\lambda}^p(E_k(r\xi)) d\sigma(\xi) = o\left(\sum_{k=1}^{\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]} 2^{p\gamma(k+1)} (1-r)^{\gamma p}\right). \quad (22)$$

Applying estimates (21) and (22) we deduce

$$\begin{aligned} & \int_S |u_2(r\xi)|^p d\sigma(\xi) \\ & \leq \frac{c_{12}(n, p)}{(1-r)^{np}} \left(\sum_{k=1}^{\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]} + \sum_{k=\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]+1}^{\infty} \right) \int_S \frac{\tilde{\lambda}^p(E_k(r\xi))}{2^{\alpha np k}} d\sigma(\xi) \\ & = \frac{o(1)}{(1-r)^{np-\gamma p}} \sum_{k=1}^{\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]} \frac{2^{p\gamma(k+1)}}{2^{\alpha np k}} \\ & \quad + \frac{c_{13}}{(1-r)^{np-\gamma p}} \sum_{k=\left[\frac{1}{2} \log_2 \frac{1}{1-r}\right]+1}^{\infty} \frac{2^{p\gamma(k+1)}}{2^{\alpha np k}} \\ & = \frac{o(1)}{(1-r)^{p(n-\gamma)}} + \frac{c_{14}}{(1-r)^{p(n-\gamma)}} \frac{1}{(1-r)^{\frac{p}{2}(\gamma-\alpha n)}} \\ & = \frac{o(1)}{(1-r)^{p(n-\gamma)}}, \quad r \uparrow 1. \end{aligned}$$

Since the integral of $|u_1|^p$ admits the same estimate, it completes the proof of Theorem 4. \square

Proof of Proposition 5. Let $\nu(B) = M > 0$. Applying Fubini's theorem (cf. the proof of Lemma 1) we deduce

$$\begin{aligned} & \int_S \nu^p(C(\xi, \delta)) d\sigma(\xi) \leq M^{p-1} \int_S \nu(C(\xi, \delta)) d\sigma(\xi) \\ & = M^{p-1} \int_S \int_{C(\xi, \delta)} d\nu(z) d\sigma(\xi) \\ & \leq M^{p-1} 2^{2n-1} \int_{1-\delta \leq |z| < 1} d\nu(z) \int_{C(z, \delta)} d\sigma(\xi) \\ & \leq c M^{p-1} 2^{2n-1} \delta^n \int_{1-\delta \leq |z| < 1} d\nu(z) = o(\delta^n), \quad \delta \downarrow 0. \end{aligned}$$

□

An alternative proof of Theorem B(1). Suppose that the assumptions of Theorem B holds. We set $d\nu(w) = (1 - |w|)^\beta d\mu(w)$. We deduce

$$\begin{aligned}\lambda(C(\xi, \delta)) &= \int_{C(\xi, \delta)} (1 - |w|)^n d\mu(w) \\ &\leq C\delta^{n-\beta} \int_{C(\xi, \delta)} (1 - |w|)^\beta d\mu(w) \leq C\delta^{n-\beta} \nu(C(\xi, \delta)).\end{aligned}$$

By Proposition 5, $\int_S \nu^p(C(\xi, \delta)) d\sigma(\xi) = o(\delta^n)$, $\delta \rightarrow 0+$. Hence,

$$\left(\int_S \lambda^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = o(\delta^{n-\beta+\frac{n}{p}}), \quad \delta \rightarrow 0+.$$

Since $n - \beta + \frac{n}{p} < 2n$, by Theorem 4 we get the assertion of Theorem B(1). □

3 Further results

Let $\Phi: [0, 1] \rightarrow [0, +\infty)$ be an increasing function such that for some $\gamma \in (0, 2n)$ and all t, δ with $0 < \delta < 1$, $0 < t\delta < 1$ we have $\Phi(t\delta) \leq t^\gamma \Phi(\delta)$. Applying the similar arguments one can show that

$$m_p(r, G_\mu) = O\left(\frac{\Phi(1-r)}{(1-r)^n}\right), \quad r \uparrow 1$$

holds if and only if

$$\left(\int_S \lambda^p(C(\xi, \delta)) d\sigma(\xi) \right)^{\frac{1}{p}} = O(\Phi(\delta)), \quad 0 < \delta < 1.$$

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